Quantum Electrodynamics and Quantum Optics

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

Final Exam

Exam duration: 180 minutes

This is a take home written exam. Please don't look at any reference material (including exercises, books, lecture videos). Any form of collaboration between students is forbidden.

1. Part A (3pt/problem) - short questions

Please answer the following questions in brief and explain the concepts. (*Estimated time: 90 min*)

1. Coherent and squeezed states are called minimum uncertainty states. What does this mean?
2. What is the wavefunction $|\psi(\vec{r})|^2$ of the coherent light field confined between two equally highly reflective mirrors? Describe it qualitatively.
3. For computing the state evolution of a quantum state, one could use either of Schrödinger or Heisenberg pictures. Describe the difference between these two pictures. Find the transformation of the creation $\hat{a}$ and annihilation $\hat{a}^\dagger$ operators under the squeezing operator $\hat{S}(\epsilon) = e^{\epsilon \hat{a}^{\dagger 2} - \epsilon^* \hat{a}^2}$, where ϵ is a complex number. You may use the Baker-Campbell-Hausdorff (BCH) lemma:
$$e^{s\hat{A}}\hat{B}e^{-s\hat{A}} = \hat{B} + s[\hat{A},\hat{B}] + \frac{s^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + ...$$
4. Write down the expression for the second order intensity auto-correlation $g^{(2)}(\tau)$, then describe how $g^{(2)}(0)$ can be used to distinguish classical and non-classical states. Give an example for a classical and non-classical state and specify their $g^{(2)}(0)$.
5. Briefly explain the macroscopic quantum model for superconductivity. Write down the definition of Josephson junction phase operator and show its effect when acting on a number state.
6. Sketch the energy level diagram of a two-level system resonantly coupled to a cavity and describe it using the system's Hamiltonian.
7. For the Rabi oscillation, described by the semi-classical model, describe qualitatively how does the time evolution of the population of the excited state depend on the light-atom detuning. Sketch graphs and assume that the atom is initially in its ground state.
8. Explain on why one needs to quantize the light field in order to explain the spontaneous emission.
9. Describe the concept of Quantum Non-demolition Measurements (QND).
Interaction of an atom dispersively coupled to a cavity is effectively described by the Hamiltonian $\hat{H}_{\text{int}} = \hbar\frac{g^2}{\Delta}\hat{\sigma}_z\hat{a}^\dagger\hat{a}$. How can this system be used to perform a *double* QND measurements on both the state of the electromagnetic field inside the cavity and quantum state of the atom.

2. Part B (8pt/problem)

Please pick 3 questions out of the following 4 and solve the corresponding exercises. In case you answer all 4 questions (completely or partially), only the first 3 will be graded. (*Estimated time: 90 min*)

Problem 1

Consider an ideal 50:50 beam-splitter (a, b are input ports and c, d are output ports).

1. Write down the scattering matrix of the system.
2. Assume the input to be a single indistinguishable photon entering at each port, $|\Psi_{\text{in}}\rangle = |1_a, 1_b\rangle$. Calculate the output state $|\Psi_{\text{out}}\rangle$, and interpret the result.

Now imagine a similar system as mentioned above, but for electrons. Unlike photons, electrons have the Fermionic nature and obey Fermionic algebra describing by anti-commutators ($\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$):

$$\{\hat{a}_\alpha, \hat{a}_\beta\} = 0 \quad , \quad \{\hat{a}_\alpha, \hat{a}_\beta^\dagger\} = \delta_{\alpha,\beta}$$

Where $\hat{a}_\alpha$ is annihilation operator of mode α.

(c) Consider the input to be a single indistinguishable **electron** at each port, $|\Psi_{\text{in}}\rangle = |1_a, 1_b\rangle$. Calculate the output state $|\Psi_{\text{out}}\rangle$, and interpret the result.

Problem 2

An arbitrary state $|\Psi\rangle$ can be projected onto the position or momentum basis states $|x\rangle$ and $|p\rangle$ to obtain position- or momentum-state wave functions $\psi(x) = \langle x|\Psi\rangle$ and $\psi(p) = \langle p|\Psi\rangle$. These wave functions are related to each other through a Fourier transform relation:

$$\psi(p) = \int \frac{dx}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \psi(x),$$

where $|\psi(x)|^2$ and $|\psi(p)|^2$ represent the probability distribution functions for position and momentum.

1. Explain why Wigner function $W(x, p)$, as a quasi-probability distribution of position and momentum, does not represent the actual probability of finding the state in (x, p).
2. Prove that one can obtain the probability distribution functions for position and momentum ($|\psi(x)|^2$ and $|\psi(p)|^2$) from the Wigner function of a pure state:

$$W(x, p) = \int \frac{du}{2\pi\hbar} e^{-ipu/\hbar} \psi^* \left(x - \frac{u}{2}\right) \psi \left(x + \frac{u}{2}\right).$$

3. Calculate the Wigner function For a momentum state

$$\psi(x) = \frac{e^{-i\mathrm{p}x/\hbar}}{\sqrt{2\pi\hbar}},$$

and from the Wigner function, calculate the momentum distribution function $\psi(p)$ and interpret the result.

Problem 3

The name 'Transmon qubit' is an abbreviation of the term transmission line shunted plasma oscillation qubit. It is closely related to a Cooper-pair box, while operating in a regime where $E_J/E_C \gg 1$. The Hamiltonian of a transmon qubit is

$$H = 4E_C\hat{n}^2 - E_J \cos\hat{\varphi},$$

where $\hat{n} = -i\left(\frac{E_J}{8E_C}\right)^{1/4}\frac{1}{\sqrt{2}}\left(\hat{a}-\hat{a}^\dagger\right)$ and $\hat{\varphi} = \left(\frac{2E_C}{E_J}\right)^{1/4}\left(\hat{a}+\hat{a}^\dagger\right)$ is the conjugate pair of position and momentum, $E_C = \frac{e^2}{2C_\Sigma}$ is the Coulomb charging energy corresponding to one electron on the total junction capacitance C_Σ, and E_J is the Josephson energy.

(a) Expand for small $\hat{\varphi}$ and show the Hamiltonian to the lowest order is $\hat{H}_0 \approx \sqrt{8E_JE_C}(\hat{a}^\dagger\hat{a} + 1/2)$. Therefore, under the lowest level of approximation, energy levels are equally spaced and anharmonicity is absent.

(b) Now we apply perturbation theory to calculate the anharmonicity defined as $\eta \equiv (E_{21} - E_{10})/\hbar$. Expand $\cos\hat{\delta}$ up to the fourth order of $\hat{\varphi}$ in the Hamiltonian, derive the corrected energy level for transmon qubit. (in terms of E_C, E_J, and state index m)

(c) Define relative anharmonicity η_r as $\eta_r \equiv \hbar\eta/E_{10}$. Show how η_r scales with E_J/E_C as $E_J/E_C \gg 1$.

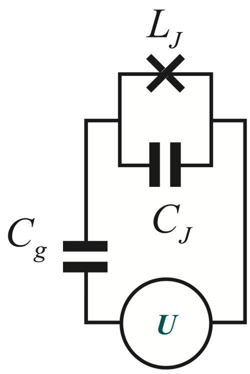

Figure 1: Circuit diagram of the transmon qubit.

Problem 4

The equivalent and simplified model of a gravitational wave detector is a cavity with one free mirror. The free mirror can be coupled to the gravitational waves and the motion of this mirror can be precisely measured via the cavity, which gives us the information about the gravitational waves. Here we are assuming the interaction picture and only work with the interaction part of the Hamiltonian:

$$\hat{H}_{\text{int}} = -\hbar g_0 \hat{a}^\dagger\hat{a}\hat{x}$$

where g_0 is the coupling strength and is a real number.

1. Assume that a strong coherent laser drive, at the frequency ω_L, is applied to the system. In the interaction picture, expand the Hamiltonian around the average coherent values of the operators e.g. $\hat{a} = \alpha + \delta\hat{a}$ and $\hat{x} = \bar{x} + \delta\hat{x}$, keeping up to quadratic terms of $\delta\hat{a}$ and $\delta\hat{x}$.

2. Using the linearized Hamiltonian, write down the equations of motion (Quantum Langevin Equation - QLE) for phase $\hat{Y}$ and amplitude $\hat{X}$ quadrature of the cavity field and the mirror position fluctuations $\delta\hat{x}$. (Cavity field quadratures are defined as $\hat{X} = \frac{\delta\hat{a}+\delta\hat{a}^\dagger}{\sqrt{2}}$ and $\hat{Y} = \frac{\delta\hat{a}-\delta\hat{a}^\dagger}{i\sqrt{2}}$)

3. Using the set of QLE that you calculated, find out which quadrature has the information about the mirror's position.